



1.6.8.6 The multiplicative inverse

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Introduction

- In this topic, we will
 - Define the multiplicative inverse of a complex number
 - Derive it from:
 - The rectangular representation
 - The application of the complex conjugate
 - The polar representation
 - Look at a geometric interpretation
 - Observe some properties





Review

- When you rationalized the denominator when the numerator was 1, you found the multiplicative inverse of the denominator:

$$\frac{1}{1+2\sqrt{3}} = \frac{1}{1+2\sqrt{3}} \cdot \frac{1-2\sqrt{3}}{1-2\sqrt{3}} = \frac{1-2\sqrt{3}}{1-12} = \frac{2\sqrt{3}-1}{11}$$

- By multiplying this by $1+2\sqrt{3}$, you see indeed that this is that multiplicative inverse:

$$\frac{2\sqrt{3}-1}{11} (1+2\sqrt{3}) = \frac{2\sqrt{3}+12-1-2\sqrt{3}}{11} = \frac{11}{11} = 1$$





0 does not have a multiplicative inverse

- To find the multiplicative inverse, we must first see that $0 + 0j$ does not have a multiplicative inverse:

- If $z = \alpha + \beta j$, then

$$\begin{aligned} 0 \cdot z &= (0 + 0j)(\alpha + \beta j) \\ &= (0 \cdot \alpha - 0 \cdot \beta) + (0 \cdot \beta + 0 \cdot \alpha)j \\ &= 0 + 0j = 0 \end{aligned}$$

- Thus, 0 multiplied by any complex number can never equal 1
 - Thus, 0 can never have a multiplicative inverse
- Thus, for z to have a multiplicative inverse:
 - At least one of α and β must be non-zero





The multiplicative inverse

- We will find the multiplicative inverse of a complex number in three different ways:
 - The rectangular representation approach
 - The complex conjugate approach
 - The polar representation approach





The rectangular representation approach

- We will start by finding it explicitly

- If $z = \alpha + \beta j$, then $z^{-1} = \gamma + \delta j$ must be such that

$$zz^{-1} = (\alpha + \beta j)(\gamma + \delta j) = 1 + 0j$$

- Thus,

$$(\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)j = 1 + 0j$$

- Two complex numbers are equal if and only if their real and imaginary components are equal, so

$$\alpha\gamma - \beta\delta = 1$$

$$\alpha\delta + \beta\gamma = 0$$





The rectangular representation approach

- Note that this is a system of two linear equations in two unknowns γ and δ , and we will swap the two equations:

$$\beta\gamma + \alpha\delta = 0$$

$$\alpha\gamma - \beta\delta = 1$$

- Assume $\beta \neq 0$, so add $-\frac{\alpha}{\beta}$ times Equation 1 onto Equation 2:

$$\beta\gamma + \alpha\delta = 0$$

$$\left(-\beta - \frac{\alpha^2}{\beta}\right)\delta = 1$$

- We can multiply Equation 2 by $-\beta$ and solve for δ :

$$\beta\gamma + \alpha\delta = 0$$

$$\delta = \frac{-\beta}{\beta^2 + \alpha^2}$$





The rectangular representation approach

- Given the system of two linear equations:

$$\beta\gamma + \alpha\delta = 0$$

$$\delta = \frac{-\beta}{\beta^2 + \alpha^2}$$

- We now perform a backward substitution of δ into Equation 1:

$$\beta\gamma + \alpha \frac{-\beta}{\beta^2 + \alpha^2} = 0$$

$$\delta = \frac{-\beta}{\beta^2 + \alpha^2}$$

- Solving this for γ , we have:

$$\gamma = \frac{\alpha}{\beta^2 + \alpha^2}$$

$$\delta = \frac{-\beta}{\beta^2 + \alpha^2}$$





The rectangular representation approach

- Thus, the multiplicative inverse of $z = \alpha + \beta j$ when $\beta \neq 0$ is

$$z^{-1} = \frac{\alpha}{\beta^2 + \alpha^2} - \frac{\beta}{\beta^2 + \alpha^2} j$$

- Rewriting this, we have:

$$z^{-1} = \frac{\alpha - \beta j}{\alpha^2 + \beta^2}$$

- You will note that this equals:

$$z^{-1} = \frac{z^*}{|z|^2}$$

- Homework: Show this also true if we assume $\alpha \neq 0$





The complex conjugate approach

- Let's look at some examples:

$$z^{-1} = \frac{\alpha - \beta j}{\alpha^2 + \beta^2}$$

- If $z = 3 + 4j$ then

$$z^{-1} = \frac{3 - 4j}{3^2 + 4^2} = \frac{3 - 4j}{25} = 0.12 - 0.16j$$

- If $z = 1 - 2j$ then

$$z^{-1} = \frac{1 + 2j}{1^2 + 2^2} = \frac{1 + 2j}{5} = 0.2 + 0.4j$$

- If $z = -0.3 - 0.1j$ then

$$z^{-1} = \frac{-0.3 + 0.1j}{0.3^2 + 0.1^2} = \frac{-0.3 + 0.1j}{0.1} = -3 + j$$





The complex conjugate approach

- Let's take a different approach:
 - “Rationalize” the denominator:

$$z^{-1} = \frac{1}{z} = \frac{1}{z} \cdot 1 = \frac{1}{z} \cdot \frac{z^*}{z^*} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2}$$

- For example,

$$\begin{aligned} (0.2 + 1.1j)^{-1} &= \frac{1}{0.2 + 1.1j} = \frac{1}{0.2 + 1.1j} \cdot \frac{0.2 - 1.1j}{0.2 - 1.1j} \\ &= \frac{0.2 - 1.1j}{1.25} = 0.16 - 0.88j \end{aligned}$$





The polar representation approach

- Let's take a final approach: if $z = |z| \angle \theta$, then $z^{-1} = r \angle \phi$ must be a complex number such that

$$\begin{aligned} zz^{-1} &= (|z| \angle \theta)(r \angle \phi) \\ &= |z|r \angle (\theta + \phi) = 1 \angle 0 = 1 \end{aligned}$$

- For this to be true, $|z|r = 1$ and $\theta + \phi = 0$
- Solving for r and ϕ , we have $r = \frac{1}{|z|}$ and $\phi = -\theta$
- Thus, $z^{-1} = \frac{1}{|z|} \angle (-\theta)$
- For example, if $z = \sqrt{2} + \sqrt{2}j = 2 \angle \frac{\pi}{4}$,
then $z^{-1} = \frac{1}{2} \angle \left(-\frac{\pi}{4}\right)$





The polar representation approach

- Let's check:
 - If $z = |z| \angle \theta$, we have two formulas for the multiplicative inverse:

$$z^{-1} = \frac{z^*}{|z|^2} \qquad z^{-1} = \frac{1}{|z|} \angle (-\theta)$$

- Are these the same?
 - Well, $z^* = |z| \angle (-\theta)$, so

$$\begin{aligned} \frac{z^*}{|z|^2} &= \frac{1}{|z|^2} (|z| \angle (-\theta)) \\ &= \frac{|z|}{|z|^2} \angle (-\theta) = \frac{1}{|z|} \angle (-\theta) \end{aligned}$$





The multiplicative inverse

- Thus, given $z = \alpha + \beta j = |z| \angle \theta$, we have that

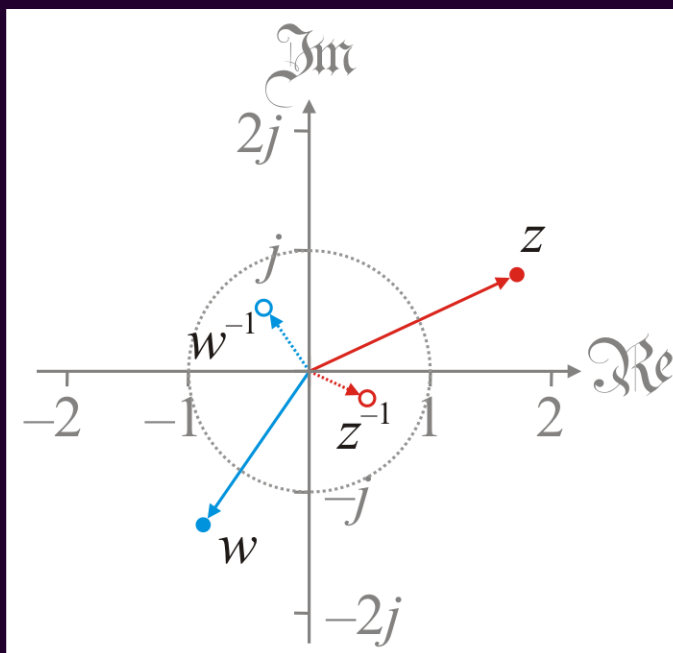
$$z^{-1} = \frac{\alpha - \beta j}{\alpha^2 + \beta^2} = \frac{z^*}{|z|^2} = \frac{1}{|z|} \angle (-\theta)$$





Geometric interpretation

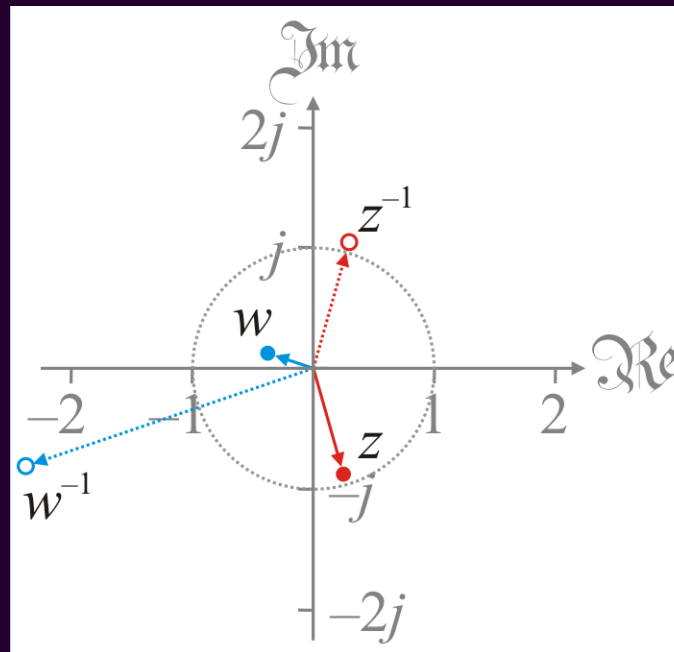
- The geometric interpretation of the multiplicative inverse is
 - The angle is that of the conjugate
 - The magnitude is one over the length of the complex number





Geometric interpretation

- Here are two examples where $|z| < 1$ and $|w| < 1$





Properties

Theorem: $(z^{-1})^{-1} = z$

Proof: $z^{-1}z = 1$ *QED*





Properties

Theorem: $(z^*)^{-1} = (z^{-1})^*$

Proof:
$$\begin{aligned} (z^{-1})^* z^* &= (z^{-1} z)^* \\ &= 1^* = 1 \quad \text{QED} \end{aligned}$$





Properties

Theorem: $|z^{-1}| = \frac{1}{|z|}$

Proof: $|z^{-1}| = \left| \frac{z^*}{|z|^2} \right| = \frac{|z^*|}{|z|^2} = \frac{|z|}{|z|^2} = \frac{1}{|z|} \quad \text{QED}$





Properties

Theorem: $z^{-1} = z^*$ if and only if $|z| = 1$

Proof: As shown, $z^{-1} = \frac{z^*}{|z|^2}$ so $\frac{z^*}{|z|^2} = z^*$ if and only if $|z|^2 = 1$,
and because $|z| \geq 0$, $|z|^2 = 1$ if and only if $|z| = 1$. *QED*





Summary

- In this topic, we've introduced the multiplicative inverse
 - We used three approaches to find three formulas
 - All three formulas are equivalent
 - We saw the geometric interpretation:
 - One over the magnitude
 - The angle of the conjugate
 - We saw a number of properties





References

- [1] https://en.wikipedia.org/wiki/Multiplicative_inverse#Complex_numbers





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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